Ch01. 1st order ODE

modeling

$2xyy' = y^2 - x^2$

$y' = \frac{y}{2x} - \frac{x}{2y}$ / $u = \frac{y}{x}$, $y' = u'x + u$

$u'x + u = \frac{u}{2} - \frac{1}{2u}$

1.4. exact ODE

$u(x,y)$, $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$

$\boxed{\frac{\partial u}{\partial x} = M, \frac{\partial u}{\partial y} = N, \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \to$ exact ODE

$u = \int M dx + k(y)$

integrating factors

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \to \frac{\partial (FM)}{\partial y} = \frac{\partial (FN)}{\partial x}$ $F(x,y)$: integrating factors

$\boxed{\frac{1}{F}\frac{dF}{dx} = R, \ R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}$

$\boxed{\frac{1}{F^*}\frac{dF^*}{dy} = R^*, \ R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}$

1.5. Linear ODEs

1) homogeneous linear ODE.

$y' + p(x)y = 0 \to y(x) = ce^{-\int p(x)dx}$

2) nonhomogeneous linear ODE.

$y' + p(x)y = r(x) \to y(x) = e^{-h}\left(\int e^h r\,dx + c\right)$, $h = \int p(x)dx$

3) Bernoulli equation

$y' + p(x)y = g(x)y^a \to$ a=0 or a=1: linear, otherwise: nonlinear

$\boxed{u' + (1-a)pu = (1-a)g \text{ : linear}}$

1.6 Orthogonal ...?

y', $\tilde{y}' = -\frac{1}{f(x,\tilde{y})}$

1.7 Initial Value Problem

Ch02. 2nd order linear ODE

$y'' + p(x)y' + q(x)y = r(x)$ (homogeneous; $r(x) = 0$)

2.1 Homogeneous

y_1, y_2 is a solution of ODE $\to y = c_1y_1 + c_2y_2$ (Linear combination) → only homogeneous linear ODE

(general solution) + (Initial value) = (Particular solution)

2.2 Homogeneous linear ODE with constant coefficients ($p(x) = a, q(x) = b$)

$y'' + ay' + by = 0 \to y = e^{\lambda x}$, $\lambda^2 + a\lambda + b = 0$

1) 2 real roots λ_1, λ_2 : $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$

2) real double root $\lambda = -\frac{a}{2}$: $y = (c_1 + c_2x)e^{\lambda x}$

3) complex roots $\lambda_1 = -\frac{a}{2} + i\omega, \lambda_2 = -\frac{a}{2} - i\omega$: $y = e^{-\frac{a}{2}x}(A\cos\omega x + B\sin\omega x)$ $e^{it} = \cos t + i\sin t$: Euler formula

2.5 Euler-Cauchy Equation

$x^2y'' + axy' + by = 0 \to y = x^m$

$m^2 + (a-1)m + b = 0$

1) 2 real roots : $y = c_1x^{m_1} + c_2x^{m_2}$

2) real double roots : $y = (c_1 + c_2\ln x)x^m$

2.6 Wronskian

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \to$ $W = 0$: y_1 & y_2 are dependent, $W \neq 0$: y_1 & y_2 are independent

2.7 nonhomogeneous ODE

$y(x) = y_h(x) + y_p(x)$

2.10 Variation Parameters

$y_p(x) = -y_1\int \frac{y_2 r}{W}dx + y_2\int \frac{y_1 r}{W}dx$

Ch03. Higher order ODEs

3.1 homogeneous linear ODEs

general/particular solution: $y = h(x)$

$k_1y_1(x) + \cdots + k_ny_n(x) = 0$

$y_1 = -\frac{k_2}{k_1}y_2$ or $y_2 = -\frac{k_1}{k_2}y_1 \to y_1, y_2$ is dependent

4th order : $y = e^{\lambda x}$, $\mu = \lambda^2$

3.2. homogeneous linear ODEs with constant coefficient

1) distinct real roots : $y = c_1e^{\lambda_1 x} + \cdots + c_ne^{\lambda_n x}$

2) simple complex roots : $y_1 = e^{\gamma x}\cos\omega x, y_2 = e^{\gamma x}\sin\omega x$

3) multiple real roots : $y(x) = c_1e^{\lambda x} + c_2xe^{\lambda x} + \cdots + c_mx^{m-1}e^{\lambda x}$

4) multiple complex roots : $y = e^{\gamma x}[(A_1 + A_2x)\cos\omega x + (B_1 + B_2x)\sin\omega x]$

3.3. nonhomogeneous linear ODEs

$y_p(x) = y_1\int \frac{W_1}{W}r\,dx + y_2\int \frac{W_2}{W}r\,dx$